The Magnetic Force as a Consequence of Special Relativity

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Part I: Introduction

Until the 1800s, scientists have thought of the electric and magnetic forces as being two separate ones. In reality, they are two sides of the same coin, connected and transformed into one another through Lorentz transformations (a change in reference frame). In this article, we explore how the magnetic force can be derived from the electric force through a relativistic setup.

Part II: The Setup

We consider two (infinite) lines of charge with linear density $+\lambda$. Put a charge $+q$ at a distance d from the line.

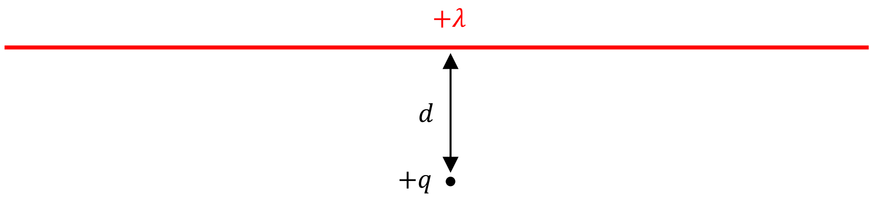

Fig. 1: A diagram showing the described setup

In the laboratory's frame, only the charge is moving. We wish to calculate the force acted on the charge, which scientists at that time knew must be purely electric, since no magnetic effects are observed.

By symmetry, the electric field must point radially away from the wire. We consider a cylinder aligned with the wire, with radius d and length $\ell \to \infty$. By Gauss's law,

$$(E)(2\pi d\ell) = \frac{\lambda\ell}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi\varepsilon_0 d}$$

$$\therefore \mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0 d}\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the radial unit vector that points directly away from the wire. Hence, the force is

$$\mathbf{F} = q\mathbf{E} = \frac{q\lambda}{2\pi\varepsilon_0 d}\hat{\mathbf{r}}$$

Part III: Enter Special Relativity

Now, we consider a more interesting approach – we consider the setup in a frame moving with a velocity v to the left. In this frame, both the point and line of charge move with a velocity v to the right. In addition to the electric force, there is now also a magnetic force acting on the particle.

Even though we supposedly don't know how to calculate the magnetic force, we know what the total force should be – we calculated it just now, its just off by a relativistic factor. Under Lorentz boosts, forces perpendicular to the boost direction transform like

$$\mathbf{F}' = \frac{\mathbf{F}}{\gamma}$$

where $\mathbf{F}'$ is the force in the boosted frame. As such, in this frame, the total force should be

$$\mathbf{F} = \frac{q\lambda}{2\pi\gamma\varepsilon_0 d}\hat{\mathbf{r}}$$

which is the sum of the electric and magnetic forces acting on the particle. We know how to calculate the electric force, it's just like in Part II. But it's a little different, due to length contraction. Consider infinitesimal point charges separated by a uniform infinitesimal distance along the line. That infinitesimal distance gets contracted, forcing those unchanged infinitesimal charges to get closer together, causing the linear density of the entire line of charge to increase. We have

$$\lambda' = \frac{\mathrm{d}q}{\mathrm{d}x'} = \frac{\mathrm{d}q}{\mathrm{d}x/\gamma} = \gamma\frac{\mathrm{d}q}{\mathrm{d}x} = \gamma\lambda$$

so the electric force acting on the particle is

$$\mathbf{F}_e' = \frac{\gamma q\lambda}{2\pi\varepsilon_0 d}\hat{\mathbf{r}}$$

The magnetic force must then be

$$\mathbf{F}_b' = \mathbf{F}' - \mathbf{F}_e = \frac{q\lambda}{2\pi\varepsilon_0 d}\left(\frac{1}{\gamma} - \gamma\right)\hat{\mathbf{r}} = -\frac{\gamma q\lambda}{2\pi\varepsilon_0 d}\frac{v^2}{c^2}\hat{\mathbf{r}}$$

Using the fact that $\varepsilon_0\mu_0 = 1/c^2$ and recognizing $\lambda' v = \gamma\lambda v = I$, the current carried by the line of charge with respect to the reference frame, we have

$$\mathbf{F}_b' = -qv\left(\frac{\mu_0 I}{2\pi d}\right)\hat{\mathbf{r}}$$

Indeed, this equation must hold true for all frames; this above equation should therefore describe the magnetic force acting on a point charge by a current-carrying wire.

Part IV: Generalisations

In general, the magnetic force acting on a point charge q is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{v}$ the velocity vector of the charge. In the setup above, $\mathbf{B}$ at the point charge is a vector pointing into the paper of magnitude $\mu_0 I/2\pi d$. Performing the cross product with the velocity vector of the charge gives a magnetic force pointing *towards* the wire.

We might want to verify our claims above. If we are given full knowledge of the mechanics of electrodynamics, we can employ Ampere's Circuital Law and consider going around a circular loop perpendicular to the direction of the current in a direction given by the right-hand rule.

$$(B)(2\pi d) = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi d}$$

with the vector pointing into the paper. As such, the magnetic force is given by

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = qv\left(\frac{\mu_0 I}{2\pi d}\right)(-\hat{\mathbf{r}})$$

matching our prediction above from special relativity.

Having discovered the equation for the magnetic force acting on a charged particle in the case of a current-carrying wire through special relativity, we have proven that the electric and magnetic forces are interrelated and just two sides of the same coin. Further developments of electromagnetism in special relativity incorporates both fields into a single object called the electromagnetic field tensor, usually denoted $F^{\mu\nu}$, which fits nicely into the language framework of four-vectors and Einstein notations. This is the first successful attempt at unification between two seemingly different forces, and physicists have made even more ambitious endeavours to unify others, for example, the electroweak force.

Part V: Conclusion

We hope you have enjoyed this article. If you have any questions or noticed that we had made a mistake (we are after all just physics enthusiasts), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at Prime Pursuit for more articles and monthly problems!