P^2SC Senior Division

May 2024

Instructions

Please read the instructions carefully. Do not turn over until you are told to do so.

- You have two hours to finish the competition.
- Write your personal information on BOTH sides of the answer sheet.
- Full marks will be given for the correct answer. Zero marks will be given for incorrect/no answers.
- Write your answer clearly with pen. There is only one correct answer for each question. Cross out any answers that you don't want submitted; we will only accept the first answer for each question.
- Do your rough work on a separate piece of paper and leave only your final answer on the answer sheet.
- The paper is designed to have a range of difficulties, and we are not expecting anyone to finish the paper.
- You will not be rewarded for showing working or partial answers, so it is more advantageous to completely solve a question rather than making partial progress on several questions.
- You may use pencil, ruler, and compasses for your rough work. Protractors, calculators, or any electronic devices are not allowed.
- Put your hand up to ask for extra rough paper or to go to the bathroom. You may NOT ask any questions about the problems.
- You can leave the competition early, but not within the first 30 or last 15 minutes. Once you leave, you are not allowed to re-enter the competition.
- You may not take the problem sheet or any rough paper away from the competition.
- Any attempt to cheat or help others cheat before, during, or after the competition will lead to disqualification.
- Do not discuss the problem until they are officially released on the Prime Pursuit website. (https://primepursuit.github.io/index.html)

P^{2}SC Senior Problems

There are 25 question in total, roughly in increasing order of difficulty. Each section indicates the number of marks per question. Good luck!

One Mark Questions

1. Find the last digit of 2024^{2024}.
2. George's basketball team so far has won 70% of their basketball games. Their team goes on a losing streak and loses their next 42 games. But then they go on a winning streak and win their next a games. It turns out that in the end George's team has won 70% of their total games. Find the value of a.
3. In the quadrilateral $ABCD$ all internal angles are less than $180°$. Suppose that $AB = AD$, $BD = BC$, $\angle BAD = \angle BCD$, and $\angle ABC = \angle ADC$. Find the value of $\angle BAD$ in degrees.
4. Jay writes "FARMOWL" on a blackboard. Each minute, he is allowed to swap the positions of two adjacent letters. What's the minimum time needed for Jay to get to the word "WOLFRAM"?
5. The number 2024 is special because its last twoy digits are 24, and 2024 itself is divisible by 23. Find the number of integers ≤ 40000 such that the last two digits are 24 and itself is divisible by 23 (including 2024).

Two Mark Questions

6. Let D, E and F be points on line segments BC, CA and AB of equilateral $\triangle ABC$ with $AB = BC = CA = 21$. Suppose that $DE \perp CA$, $EF \perp AB$, and $FD \perp BC$. Find the length of AE. *(The $\perp$ symbol means that two lines are perpendicular to each other.)*
7. Let S be the set of numbers such that it has exactly 3 one's and at most 5 digits in its binary representation. Find the sum of elements of S in base 10. *(E.g. The number 100101 written in the normal base 10 is equal to $2^5 + 0 + 0 + 2^2 + 0 + 2^0 = 32 + 4 + 1 = 37$.)*
8. Given that $x^2 + xy + y = 2024, y^2 + xy + x = 46$, find the sum of all possible values of $x + y$.
9. $\triangle ABC$ is right angled at B. D is a point on line AC such that $DA = AC$ and D is distinct from C. Given that $AB = 3$ and $BC = 8$, find the length of BD.
10. Find the number of 3 digit integers (with non-zero starting digit) such that the first digit is divisible by two, the sum of the first 2 digits is divisible by 3, and the sum of all digits is divisible by 5.

Three Mark Questions

11. Michael places 36 points evenly around a circle. How many ways can he choose 3 different points so that in the triangle formed by those 3 points, at least one of the angles is $60°$?

12. In triangle $ABC, \angle BCA = 90°$. D is a point on AB such that CD is perpendicular to AB. Given that $[BDC]^2 = [ABC][ADC]$, find the value of $\sin \angle ABC$. *(Here* $[ABC]$ *denotes the area of* $\triangle ABC$*)*

13. For a positive integer n, let P_n denote the product of all positive divisors of n (including 1 and n). Find biggest $n \leq 2048$ such that $\log_n(P_n)$ is not an integer.

14. The Golden Ratio, φ, satisfies the equation $\varphi^2 = \varphi + 1$. Some expressions involving φ can be re-written in the form $a\varphi + b$. *E.g.* $\varphi^2 + \varphi = \varphi + 1 + \varphi = 2\varphi + 1$.

    Consider the following expression:

    $$\frac{\varphi^4 + 1}{1 - \varphi} + \frac{\varphi^4 + 1}{(1 - \varphi)^2} + \frac{\varphi^4 + 1}{(1 - \varphi)^3} + \dots.$$

    It can be be written in the form $a\varphi + b$ for rational numbers a and b. Find the value of $a + b$.

15. Sharon chooses a set C of 50 distinct positive integers. For each positive rational number, she gets one coin if it can be written in the form $\frac{a}{b}$, where a and b are numbers in C. Let M be the maximum number of coins, and m be the minimum number of coins she can obtain across all possible choices of the set C. Find the value of $M + m$.

Four Mark Questions

16. Helen draws a regular 2023-gon. She labels each vertex of the polygon with the number 1 or -1. Then for each possible diagonal (including edges) of the polygon, she labels it with the product of the vertex labels at the ends of the line segment. Finally, she calculates the sum of all edge labels. Let N be the minimum *positive* value that Helen's score can be, and let p be the number of vertices labeled 1 when she achieves this score. Find the value of $N + p$.

17. A penguin moves in straight lines within an equilateral triangle, while drawing a line along its trail. It starts at S and travels horizontally. Each time it touches a line (one of the original big triangle or a line it drew), it bounces off at the same angle. The penguin stops after it reaches the intersection of 2 lines. Find the total distance travelled by the penguin. (*See next page for the diagram.*)

18. The numbers $1, 2, \dots, 52$ are written on a whiteboard. Alston splits the 52 numbers into 26 pairs of two. For each pair of integers, he replaces the bigger one with the smaller one plus 1. After doing this to all 26 pairs, he finds S, the sum of all 52 resulting integers. Find the difference between the maximum and minimum values of S across all possible initial pairings.

19. Let $ABCD$ be a square. Let E be a point on side DC. Draw square $EBGF$ such that F and A are opposite sides of the line BE. Extend DC to a point H on GF. Given that $\frac{GH}{EH} = \frac{2024}{2023}$, find $\frac{DE}{FE}$.

20. The polynomial $f(x) = x^3 + ax^2 + bx + 2024$ has roots p^2, q^2, r^2 for some possibly complex and not necessarily distinct numbers p, q, r. The monic polynomial $g(x)$ has roots $\frac{pq}{r}, \frac{qr}{p}, \frac{rp}{q}$. Suppose that $f(1) = g(1)$. Find the value of a.

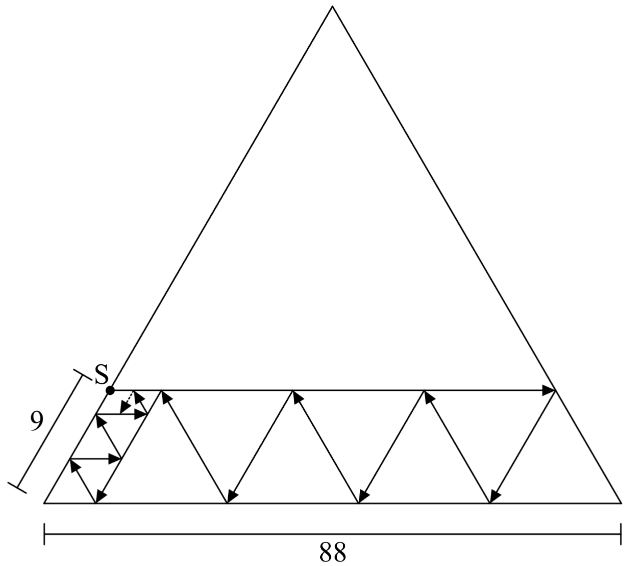

Figure 1: NOT TO SCALE

Five Mark Questions

21. $\triangle ABC$ satisfies $AB = AC, BC = 2024$. Let D be a point on segment AC such that D does not coincide with A or C. Let E be the incenter of $\triangle ABD$, and BD intersects AE at F. The circumcircles of $\triangle ABD$ and $\triangle DFC$ intersects at a point P other than D. Given that $PE = 1012\sqrt{2}$, find the value of $\angle EBC$ in degrees. *(The incenter of $\triangle ABC$ is the center of the inscribed circle, or incircle, or $\triangle ABC$. The circumcircle of $\triangle ABC$ is the circle passing through A, B and C.)*

22. 100 people stand in a circle all facing inwards. Each person raises either their left hand, right hand or both sideways. If two adjacent people have their hands extended towards each other, they hold hands. Find the expected value for the number of pairs of hands held in total.

23. Given that complex numbers x, y, z satisfy $x^2 = y^2 + z - 1, y^2 = z^2 + x - 1, z^2 = x^2 + y - 1$, find the sum of all possible values of $x^2 + y^2 + z^2$.

24. Let x and y be two positive integers satisfying

$$xy - (x + y) = \gcd(x, y) + \operatorname{lcm}(x, y)$$

where $\gcd(x, y)$ and $\operatorname{lcm}(x, y)$ are respectively the greatest common divisor and the least common multiple of x and y. Find the maximum possible value of $x + y$.

25. Let H be the orthocenter of acute triangle ABC. Let r and R be the inradius and circumradius of $\triangle ABC$ respectively. Given that $AH + BH + CH = 2024$, find the value of $r + R$. *(The orthocenter of $\triangle ABC$ is the intersection of its three altitudes. The inradius and circumradius of $\triangle ABC$ is the radius of its incircle and circumcircle respectively.)*