P^2SC Junior Division

May 2024

Instructions

Please read the instructions carefully. Do not turn over until you are told to do so.

- You have two hours to finish the competition.
- Write your personal information on BOTH sides of the answer sheet.
- Full marks will be given for the correct answer. Zero marks will be given for incorrect/no answers.
- Write your answer clearly with pen. There is only one correct answer for each question. Cross out any answers that you don't want submitted; we will only accept the first answer for each question.
- Do your rough work on a separate piece of paper and leave only your final answer on the answer sheet.
- The paper is designed to have a range of difficulties, and as a result we are not expecting anyone to finish the paper. You will not be rewarded for showing working or partial answers. Therefore it is more advantageous to completely solve a question rather than making partial progress on several questions.
- You may use pencil, ruler, and compasses for your rough work. Protractors, calculators, or any electronic devices are not allowed.
- Put your hand up to ask for extra rough paper or to go to the bathroom. You may NOT ask any questions about the problems.
- You can leave the competition early, but within the first 30 or last 15 minutes. Once you leave, you are not allowed to re-enter the competition.
- You may not take the problem sheet or any rough paper away from the competition.
- Any attempt to cheat or help others cheat before, during, or after the competition may lead to disqualification.
- Do not discuss the problem until they are officially released on the Prime Pursuit website (https://primepursuit.github.io/index.html).

P^2SC Junior Problems

There are 25 question in total, roughly in increasing order of difficulty. Each section indicates the number of marks per question. Good luck!

One Mark Questions

1. George and his friends went to Hobbiton and spent 288 dollars in total. They agreed to share their expenses equally. However George had no money to pay his share, and each of his friends had to pay 4 dollars more to cover all the expenses. How many friends does George have?
2. Phoenix's birthday is such that the product of the month and date is equals to 308. Find the sum of the month and date of her birthday.
3. In the quadrilateral $ABCD$ all internal angles are less than $180°$. Suppose that $AB = AD$, $BD = BC$, $\angle BAD = \angle BCD$, and $\angle ABC = \angle ADC$. Find the value of $\angle BAD$ in degrees.
4. A cuboid has dimensions 8cm by 10cm by 12cm. It is made up of separate cubic centimeter cubes. If the cuboid is dipped in paint and then taken apart, what is the probability that any cube chosen at random has no paint on any sides?
5. If the sum of two numbers 45 and $\dfrac{45}{S}$ is equal to its product, find the value of S.

Two Mark Questions

6. What is the smallest 3 digit number such that it has a remainder of 1 when divided by 3, remained of 3 when divided by 5, and a remainder of 5 when divided by 7?
7. For real numbers a and b, define $f(n) = a^n + b^n$ for all positive integers n such that $f(3) = [f(1)]^3 + f(1)$. Find the value of ab.
8. Let D, E and F be points on line segments BC, CA and AB of equilateral $\triangle ABC$ with $AB = BC = CA = 21$. Suppose that $DE \perp CA$, $EF \perp AB$, and $FD \perp BC$. Find the length of CD. *(The $\perp$ symbol means that two lines are perpendicular to each other.)*
9. Jay writes "FARMOWL" on a blackboard. Each minute, he is allowed to swap the positions of two adjacent letters. What's the minimum time needed for Jay to get to the word "WOLFRAM"?
10. Given that $x^2 + xy + y = 2024$ and $y^2 + xy + x = 46$, find the sum of all possible values of $x + y$.

Three Mark Questions

11. $ABCD$ is a parallelogram such that $AB = 4, AD = 3, \sin A = 2/3$. Points E and F are on AB and BD such that $EF \parallel AD$, and the area of $EBCF$ is 3. Find the value of AE. *(The $\parallel$ symbol means that two lines are parallel to each other.)*

12. Michael places 22 points evenly around a circle. How many ways can he choose 3 different points so that they form a right-angled triangle?

13. Given that the function $g(x)$ has 11 distinct roots, and that $g(28 - x) = g(28 + x)$ for all real values of x, find the sum of all the roots of $f(x)$. *(A root of a function is a solution to the equation $g(x) = 0$.)*

14. Let $\lfloor x \rfloor$ be the largest integer not exceeding x. For example, $\lfloor 6 \rfloor = 6$ and $\lfloor 3.14 \rfloor = 3$. Find the value of $\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \cdots + \lfloor \sqrt{99} \rfloor$.

15. Find the number of ways of choosing 2 different numbers from the 50 positive integers $1, 2, 3, \ldots, 50$ such that the sum of these two numbers is not less than 50.

Four Mark Questions

16. In triangle $ABC, \angle BCA = 90°$. D is a point on AB such that CD is perpendicular to AB. Given that $[BDC]^2 = [ABC][ADC]$, find the value of $\sin \angle ABC$. *($[ABC]$ denotes the area of $\triangle ABC$)*

17. George selects n numbers from $1, 2, ..., 2024$ such that the difference of any two numbers is not a prime. Find the maximum value of n.

18. Call a 4-digit number *exhilarating* if each of its digits are either 1,2,3 or 4 (repetitions are allowed) Find the sum of all exhilarating numbers.

19. Let $ABCD$ be a square. Let E be a point on side DC. Draw square $EBGF$ such that F and A are opposite sides of the line BE. Extend DC to a point H on GF. Given that $\dfrac{GH}{EH} = \dfrac{2024}{2023}$, find $\dfrac{DE}{FE}$.

20. Let f be a function that takes positive integer inputs. It is given that $f(1) = 1, f(2) = 0$, and for $n \geq 2$:

$$f(2n) = 2f(n)$$
$$f(2n-1) = f(n) + f(n-1).$$

E.g. $f(7) = f(4) + f(3) = 2f(2) + f(2) + f(1) = 1$. Find the number of positive integers $n \leq 1024$ such that $f(n) \geq 24$.

Five Mark Questions

21. A penguin moves in straight lines within an equilateral triangle with side lengths of 88m, while drawing a line along its trail. It starts by travelling horizontally starting 9m up one of the non horizontal sides. Each time it touches a line (one of the original big triangle or a line it drew), it bounces off at the same angle. The penguin stops after it reaches the intersection of 2 lines. Find the total distance travelled by the penguin.

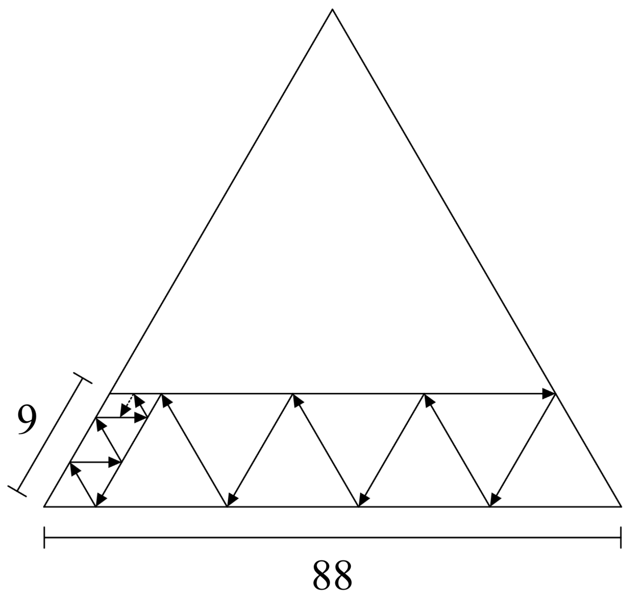

Figure 1: NOT TO SCALE

22. Find the value of

$$\frac{1}{7^1} + \frac{1}{7^2} + \frac{2}{7^3} + \frac{3}{7^4} + \frac{5}{7^5} + \frac{8}{7^6} + \frac{13}{7^7} + \dots$$

where the numerators are the Fibonacci numbers (its convergence is given).

23. If P is an arbitrary point in the interior of a scalene triangle ABC, find the probability that the area of $[\triangle ABP] > [\triangle ACP] > [\triangle BCP]$.

24. Hayden the ant is on a vertex of a cube. Every minute, he randomly walks (with equal probability) to one of the 3 adjacent vertices. What's the probability that he ends up where he started after exactly 6 minutes?

25. Given that $\dfrac{x}{y+z} + \dfrac{y}{z+x} + \dfrac{z}{x+y} = 1$, find the value of $\dfrac{x^2}{y+z} + \dfrac{y^2}{z+x} + \dfrac{z^2}{x+y}$.