NZSST Discussion 2024

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Problems by the NZMOC

January 27, 2024

Recently the NZ Squad Selection Test was held over a three-day period in mid-January, 2024. While I enjoyed most of the problems, I'd like to discuss four which I found quite interesting. Hopefully you could learn a thing or two, and at the same time, enjoy the process of problem solving throughout.

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1 Problems

1.1 NZSST3 2024 P7: Trial and Error

Given that $0 \leq a, b, c \leq 1$, what is the maximum possible value of the following expression?

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1}$$

Solution Outline: Traditionally one would guess the maximum is achieved when $a = b = c = 1$, where the expression would be equal to $\frac{3}{2}$. However, upon further inspection one would notice $a, b, c = 1, 1, 0$ yields a maximum value of 2. Intuitively this somewhat makes sense, as one could try to maximise $\frac{a}{bc+1}$ or $\frac{a}{bc+1}$ by minimising the denominator. You could play around with the other values of (a, b, c), but for now, let's conjecture

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1} \leq 2.$$

Since inequality does not hold at $a = b = c$, it would be more sensible to deal with the problem by each parts. First, we will try to prove

$$\frac{a}{bc+1} \leq \frac{2a}{a+b+c}.$$

This is equivalent to $a + b + c \leq 2bc + 2$, $(a \neq 0)$. Now, we will make use of the fact that $0 \leq a, b, c \leq 1$, rewriting the inequality as

$$(1-b)(1-c) + bc + (1-a) \geq 0$$

which is true since $1-a$, $1-b$, $1-c$ and bc are all nonnegative. By symmetry, we have

$$\frac{b}{ac+1} \leq \frac{2b}{a+b+c}$$
$$\frac{c}{ab+1} \leq \frac{2c}{a+b+c},$$

which implies

$$\frac{a}{bc+1} + \frac{b}{ac+1} + \frac{c}{ab+1} \leq \frac{2(a+b+c)}{a+b+c} = 2.$$

Equality is achieved when $(a, b, c) = (1, 1, 0)$. Q.E.D.

Comments: Now why is 2 the maximum? Suppose the maximum is $2 + 3n$ for some positive n. Let's follow the orignal method, striving to prove

$$\frac{a}{bc+1} \leq \frac{2a}{a+b+c} + n$$
$$\Leftrightarrow \quad a + b + c \leq 2bc + 2 + \frac{n}{a}(a+b+c)(bc+1)$$
$$\Leftrightarrow \quad 0 \leq (1-b)(1-c) + bc + (1-a) + \frac{n}{a}(a+b+c)(bc+1).$$

But $\frac{n}{a}(a+b+c)(bc+1) \geq \frac{n}{a} \cdot a = n > 0$ by assumption, hence equality could never be attained. This establishes 2 as the maximum, which is our final answer.

1.2 NZSST3 2024 P8: The Hammer and the Fly

Find all triples (a, b, c) of positive integers such that both $\frac{b+1}{a}$ and $\frac{a^2+1}{bc+1}$ are integers.

Solution Outline: Some people would notice the similarity of this problem with the legendary 1988 IMO P6:

(IMO 1988 P6) *Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that*

$$\frac{a^2+b^2}{ab+1}$$

is the square of an integer.

Here the more advanced technique of Vieta Jumping was used. However, my philosophy of problem solving is to first try simple methods and intuitive ideas, as opposed to ”killing a fly with a hammer” by using advanced theorems. Moreover, the abundance of variables and the non-symmetrical nature of the expression does not make it easy for us to use Vieta Jumping. With that in mind, let $b = ak - 1$ for some positive integer k, then

$$\begin{aligned} bc + 1 &= (ak-1)c + 1 \\ &= ack - c + 1. \end{aligned}$$

Let M be an integer such that $M = \frac{a^2+1}{bc+1}$, then

$$\begin{aligned} (bc+1)M &= a^2 + 1 \\ (ack - c + 1)M &= a^2 + 1. \end{aligned}$$

I hope up to this point everything is intuitive. Now, the key observation:

$$a(ckM - a) = cM - M + 1.$$

First, we introduce a lemma.

Lemma: For positive integers x,y and n, if $xy = n$, then $x + y \leq n + 1$.

Proof: There are two cases to consider.

Case 1: $\{x, y\} = \{n, 1\}$, then $x + y = n + 1$.

Case 2: $x, y \neq n$, then $x, y \leq \dfrac{n}{2}$, hence $x + y \leq \dfrac{n}{2} + \dfrac{n}{2} = n < n + 1$.

Using this lemma with $(x, y, n) = (a, ckM - a, cM - M + 1)$ gives us

$$ckM \leq cM - M + 2.$$

With the above inequality in mind, now we shall resort to case "bashing".

If $k = 1$, then $M = 1$ or 2. The former gives the solution $(a, b, c) = (2, 1, 4)$, while the latter gives the family of solutions

$$\{(2m - 1, 2m - 2, m) \,|\, m \in \mathbb{Z}^+,\, m \geq 2\}.$$

If $k = 2$, then

$$\begin{aligned} M(c + 1) &\leq 2 \\ \Rightarrow M &= c = 1 \end{aligned}$$

which gives the solution $(1, 1, 1)$. (The details are left as an exercise for the reader.)

If $k \geq 3$, then $ckM \geq 3cM$. However, this implies

$$\begin{aligned} cM - M + 2 &\geq 3cM \\ 2 &\geq M(2c + 1) \end{aligned}$$

which yields no solutions as $M(2c + 1) \geq 3$.

Hence the only answers are $\{(2m - 1, 2m - 2, m) \,|\, m \in \mathbb{Z}^+,\, m \geq 2\}$, $(2, 1, 4)$, and $(1, 1, 1)$. These can be easily verified.

Comments: As you can see, the use of Vieta Jumping is not really necessary, as shown above. "Keeping it simple, and don't overthink", as my friend Alex Chui said before, is always a good motto to remember when dealing with problems.

1.3 NZSST2 2024 P6: A Bit of Everything

Given are two triangles A and B, both having area a and perimeter p. Prove that there exists a triangle C such that C is congruent to B, and such that the intersection of the interiors of A and C is a polygon with area greater than

$$\frac{13a^2}{p^2}.$$

Solution Outline: At first glance the problem might seem "undoable"; 13 seems like an arbitary number, and the area of the intersecting area doesn't seem easy to handle. Nevertheless, let's try manipulating the seemingly out of place expression, using the fact that $\frac{pr}{2} = a$ (where r is the inradius.)

$$\frac{13a^2}{p^2} = \frac{13r^2}{4}$$

Now notice that equal area and perimeter implies the two triangles have equal inradius (a fact that motivates this solution.) This hints we should construct the diagram as show below, overlapping the incircles of A and C with center I.

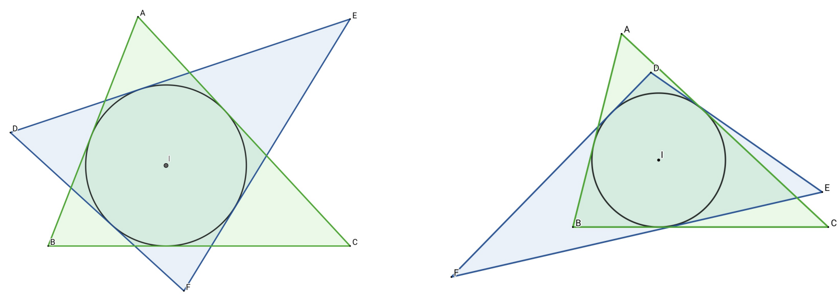

Figure 1: Note that in both diagrams the intersection of A and C forms a hexagon.

Here we constructed so that the intersecting area is a hexagon. If one of the vertices of the hexagon coincides with a vertex of A or C (forming a polygon with less than six sides), we could simply rotate C about I, since there are infinitely many rotations one could do, but only a finite number of rotations such that a hexagon isn't formed. It is clear that

$$\sum_{i=1}^{6} 2\theta_i = 2\pi \text{ (laws of tangents)}$$
$$\sum_{i=1}^{6} \theta_i = \pi.$$

Hence we could express the intersecting are of A and C, which we call S, as

$$\begin{aligned} S &= 2\sum_{i=1}^{6} \frac{1}{2} r \cdot r \tan\theta_i \\ &= \sum_{i=1}^{6} r^2 \tan\theta_i \\ &= r^2 \sum_{i=1}^{6} \tan\theta_i. \end{aligned}$$

Now note that $\theta_i \in (0, \frac{\pi}{2})$ for $i \in \{1, 2, 3, 4, 5, 6\}$, where the function $f(x) = \tan x$ is convex. This means we

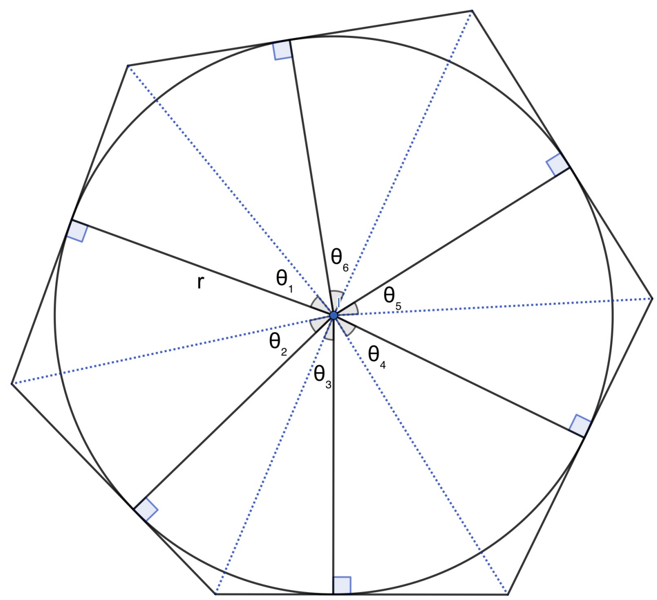

Figure 2: One can compute S by subdiving the hexagon into right-angled triangles.

can apply Jensen's inequality, giving us

$$\begin{aligned} S &= r^2 \sum_{i=1}^{6} \tan\theta_i \\ &\geq 6r^2 \tan\left(\frac{1}{6}\sum_{i=1}^{6}\theta_i\right) \\ &= 6r^2 \tan\left(\frac{\pi}{6}\right) \\ &= 6r^2 \cdot \frac{\sqrt{3}}{3} \\ &= 2\sqrt{3}r^2. \end{aligned}$$

Clearly $2\sqrt{3} > \frac{13}{4}$, which completes our proof.

Comments: The results shown generalises to all pairs of triangles with equal inradii.

1.4 NZSST3 2024 P6: Always a Sweet Treat

Triangle ABC has incenter I and satisfies $AB < AC$. Let M be the midpoint of BC, and let D be the point where the incircle of triangle ABC is tangent to side BC. The circle with center M and radius MD intersects line AI at P and Q. Prove that $\angle BAC + \angle PMQ = 180^\circ$.

This is by far the most interesting geometry problem out of all three selection tests.

(NZSST1 2024 P3) *Let ABC be a triangle with $AB = 360$, $BC = 240$ and $AC = 180$. The internal and external bisectors of angle $\angle CAB$ meet line BC at points P and Q respectively. Find the radius of the circumcircle of $\triangle APQ$.*

(NZSST2 2024 P2) *Let ABC be an acute triangle and let D be the midpoints of side BC. Suppose that $\angle BAD = \angle ACD$ and $\angle DAC = 15^\circ$. Determine $\angle ACB$.*

(NZSST3 2024 P4) *Let P be a point inside square $ABCD$. Prove that the perpendiculars from A, B, C and D to lines BP, CP, DP and AP respectively are concurrent.*

Solution Outline: It is quite logical to convert the statement into something simpler, hence we shall start with that.

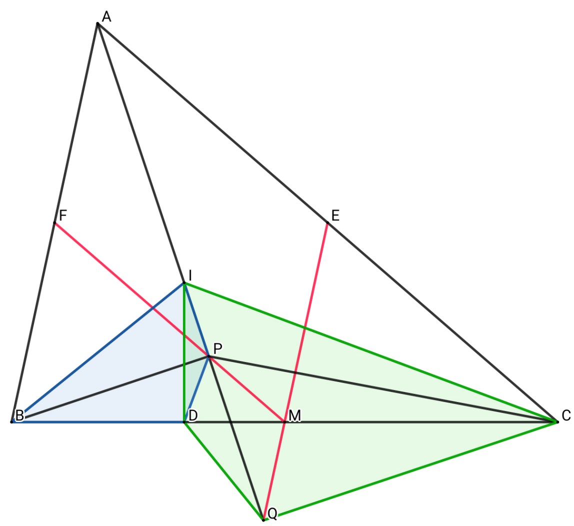

Figure 3: Observing two possible cyclic quadrilaterals and collinearities.

$$\begin{aligned}
& \angle BAC + \angle PMQ = 180^\circ \\
\Leftrightarrow \quad & \angle BAC + 360^\circ - 2\angle PDQ = 180^\circ \qquad \text{(angles in a circle)} \\
\Leftrightarrow \quad & \angle BAC + 180^\circ = 2\angle PDQ \\
\Leftrightarrow \quad & 2(\frac{1}{2}\angle BAC + 90^\circ) = 2\angle PDQ \\
\Leftrightarrow \quad & \angle BIC = \angle PDQ.
\end{aligned}$$

This simple condition allows the acute eye (and a good diagram) to make a key conjecture: $\triangle BIC \sim \triangle PDQ$.

This again is equivalent to

$$
\begin{array}{lll}
 & \triangle BIC \sim \triangle PDQ & \\
\Leftrightarrow & \begin{cases} \angle DPQ = \angle IBC \\ \angle DQP = \angle BCI \end{cases} & \text{(corr. angles in sim. triangle)} \\
\Leftrightarrow & \begin{cases} B, D, P, I \quad \text{concyclic} \\ I, D, Q, C \quad \text{concyclic} \end{cases} & \text{(substanded angles eq.)} \\
\Leftrightarrow & \angle BPA = \angle BQC = 90^\circ. &
\end{array}
$$

Proving this would essentially end the solution. However, directly attempting this does not seem simple, as the conditions on hand are somewhat "limited."

Hence we shall proceed with reverse reconstruction. Let P', Q' be the foot B, C to AI respectively. Let E, F be the midpoints of AC, AB respectively. We shall prove that P' coincides with P and Q' coincides with Q.

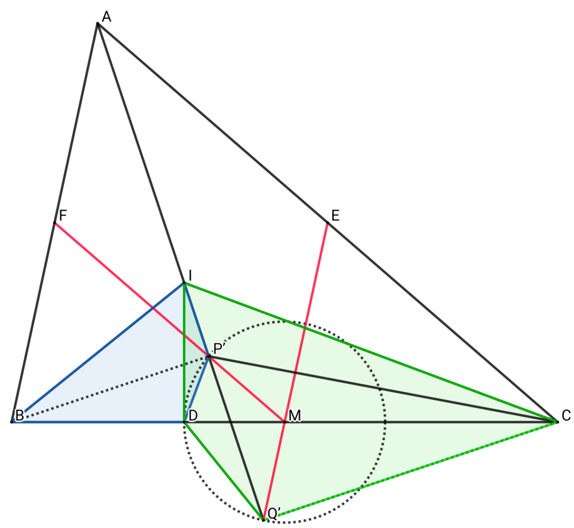

Figure 4: Reverse reconstruction to utilize these conditions to its fullest extent.

Directly one could infer

$$
\begin{array}{lll}
 & AE = EQ' & \text{(from Thale's Theorem)} \\
\Leftrightarrow & \angle Q'AE = \angle EQ'A = \angle Q'AB & \\
\Leftrightarrow & Q'E \parallel AB. &
\end{array}
$$

Since M, E are midpoints, this implies $ME \parallel AB$. Hence Q', M, E collinear.

Similarly, P', M, F are collinear (details left as an exercise for the reader.) Hence $\angle P'MD = \angle ACB$, $\angle DMQ' = \angle ABC$.

Also directly one could infer

$$\begin{cases} B, D, P', I & \text{concyclic} \\ I, D, Q', C & \text{concyclic} \end{cases}$$

as discussed. Therefore

$$\begin{aligned} \angle DP'M &= \angle DP'Q' + \angle Q'PM \\ &= \angle IBC + \angle IAC && \text{(angles in cyclic quad. and parallel lines)} \\ &= \frac{1}{2}(\angle ABC + \angle BAC) \\ &= \frac{1}{2}(180^\circ - \angle ACB). \end{aligned}$$

This implies

$$\begin{aligned} \angle P'DM &= 180^\circ - \angle ACB - \frac{1}{2}(180^\circ - \angle ACB) && \text{(sum of angles of a triangle)} \\ &= \frac{1}{2}(180^\circ - \angle ACB) \\ &= \angle DP'M. \end{aligned}$$

which means $DP' = P'M$, thus $P' = P$. One could analogously prove $Q' = Q$, and again, the details are left for the reader to complete.

2 Afterword

I hope you have enjoyed this article. If you have any questions or I have made any mistakes (I am after all just a maths enthusiast), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at *Prime Pursuit* for more articles and monthly problems!