Georg Cantor, Robert Simson, and the Complex Plane

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Simson's Theorem is a well-known configuration in the world of Olympiad Geometry. Cantor's series of Geometry Theorems built on Simson's endeavours, exploring intersections of different Cantor Lines. While it is totally plausible to prove them synthetically, the use of complex plane would be much more efficacious. In this article we will discuss the complex properties of the Simson Line, and eventually proving the mesmerizing theorems of the great Cantor.

Contents

1 Simson Line and Complex Numbers

(Simson's Theorem) Let P be a point on the circumcircle of $\triangle ABC$. If D, E, F be the feet of the perpendiculars from P to BC, CA, AB respectively, then D, E, F are collinear.

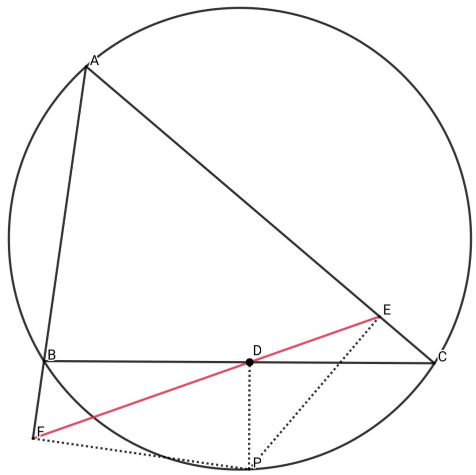

Figure 1: In the famous *Simson Line* configuration, D, E, F collinear.

Exercise 1.1 Prove Simson's Theorem using complex numbers.

Firstly, let the circumcircle of $\triangle ABC$ be the unit circle, hence $|a| = |b| = |c| = |p| = 1$. Using Lemma 6.1 (see *Appendix*) on $(a, b, z) = (b, c, p)$ gives

$$d = \frac{(\overline{b} - \overline{c})p + (b - c)\overline{p} + \overline{b}c - b\overline{c}}{2(\overline{b} - \overline{c})}$$

$$= \frac{p}{2} + \frac{b - c}{2\left(\frac{1}{b} - \frac{1}{c}\right)}\left(\frac{1}{p} - \frac{1}{b}\right)$$

$$= \frac{p}{2} - \frac{bc}{2p} + \frac{b + c}{2}$$

Similarly applying the lemma to (a, b, p) and (c, a, p) gives

$$e = \frac{p}{2} - \frac{ac}{2p} + \frac{a + c}{2}$$

$$f = \frac{p}{2} - \frac{ab}{2p} + \frac{a + b}{2}$$

Now notice that

$$d - e = \left(\frac{p}{2} - \frac{bc}{2p} + \frac{b+c}{2}\right) - \left(\frac{p}{2} - \frac{ac}{2p} + \frac{a+c}{2}\right)$$

$$= \frac{(a-b)c}{2p} - \frac{a-b}{2}$$

$$= \frac{(a-b)(c-p)}{2p}$$

Similarly,

$$d - f = \frac{(a-c)(b-p)}{2}$$

Hence,

$$\frac{d-e}{d-f} = \frac{(a-b)(c-p)}{(a-c)(b-p)}$$

However by lemma 6.2 $\dfrac{(a-b)(c-p)}{(a-c)(b-p)}$ is real $(a, c, b, p$ cyclic), hence d, e, f collinear by lemma 6.3.

Exercise 1.2 Find the complex equation of the Simson Line.

By lemma 6.4 the complex equation for the Simson Line is

$$\overline{\alpha} z - \alpha \overline{z} = \overline{\alpha} d - \alpha \overline{d}$$

where $\alpha = d - e$, now notice

$$\overline{\alpha} = \overline{\left(\frac{(a-b)(c-p)}{2p}\right)}$$

$$= \frac{\left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{c} - \frac{1}{p}\right)}{\frac{2}{p}}$$

$$= \frac{(b-a)(p-c)p}{2abcp}$$

$$= \frac{(a-b)(c-p)}{2abc}$$

and that

$$\overline{d} = \overline{\left(\frac{p}{2} - \frac{bc}{2p} + \frac{b+c}{2}\right)}$$

$$= \frac{1}{2p} - \frac{p}{2bc} + \frac{\frac{1}{b}+\frac{1}{c}}{2}$$

$$= \frac{1}{2p} - \frac{p}{2bc} + \frac{b+c}{2bc}$$

$$= \frac{1}{2p} + \frac{b+c-p}{2bc}$$

Substituting those expressions into the equation gives

$$\frac{(a-b)(c-p)}{2abc}z - \frac{(a-b)(c-p)}{2p}\overline{z} = \frac{(a-b)(c-p)}{2abc}\left(\frac{p}{2} - \frac{bc}{2p} + \frac{b+c}{2}\right) - \frac{(a-b)(c-p)}{2p}\left(\frac{1}{2p} + \frac{b+c-p}{2bc}\right)$$

Multiplying both sides by $\frac{2abcp}{(a-b)(c-p)}$ gives

$$pz - abc\overline{z} = p\left(\frac{p}{2} - \frac{bc}{2p} + \frac{b+c}{2}\right) - abc\left(\frac{1}{2p} + \frac{b+c-p}{2bc}\right)$$

Now note that

$$p\left(\frac{p}{2} - \frac{bc}{2p} + \frac{b+c}{2}\right) - abc\left(\frac{1}{2p} + \frac{b+c-p}{2bc}\right)$$

$$= \frac{1}{2p}[p^3 - bcp + (b+c)p^2] - \frac{1}{2p}[abc + a(b+c-p)p]$$

$$= \frac{1}{2p}[p^3 + (a+b+c)p^2 - (ab+bc+ac)p - abc]$$

Hence the complex equation of the Simson Line is

$$pz - abc\overline{z} = \frac{1}{2p}(p^3 + \sigma_1 p^2 - \sigma_2 p - \sigma_3)$$

where $\sigma_1 = a+b+c$, $\sigma_2 = ab+bc+ac$, $\sigma_3 = abc$.

Comments: I suggest readers be well-versed with complex numbers formulae while having solid fundamentals in algebraic manipulations, as some might find the next chapters confusing if one's not skillful enough in algebra.

With these in mind, now lets tackle the Cantor's theorems.

2 Cantor's Line

(Cantor's Line) A, B, C, D, M, N lie on a circle. The Simson Lines of M, N with respect to $\triangle BCD, \triangle ACD, \triangle ABD, \triangle ABC$ intersect at T_a, T_b, T_c, T_d respectovely. Prove that T_a, T_b, T_c, T_d collinear.

Exercise 2.1 Prove the exsistence of the Cantor Line.

Firstly, I would like to encourage the reader to actually "do the bash" as the details are quite complicated. Secondly, since we are mainly "bashing" and that diagrams really will not do us any favor, they will not be included in this article, but readers are more than welcomed to draw their own if it could provide a sense of clarity. That said, WLOG let the circumcircle of $\triangle ABC$ be the unit circle. Using our results in Section 1 we could find the equations of the Simson Lines of Lines of M, N with respect to $\triangle BCD$.

$$mz - bcd\overline{z} = \frac{1}{2m}(m^3 + \sigma_1 m^2 - \sigma_2 m - \sigma_3)$$

$$nz - bcd\overline{z} = \frac{1}{2n}(n^3 + \sigma_1 n^2 - \sigma_2 n - \sigma_3)$$

Solving for z would give us T_a. Subtracting both equations gives

$$(m-n)z = \frac{1}{2}[(m^2 - n^2) + \sigma_1(m-n) - \sigma_3\left(\frac{1}{m} - \frac{1}{n}\right)]$$

$$(m-n)z = \frac{1}{2}[(m-n)(m+n) + \sigma_1(m-n) - \sigma_3\left(\frac{n-m}{mn}\right)]$$

$$z = \frac{1}{2}\left(m + n + \sigma_1 + \frac{\sigma_3}{mn}\right)$$

$$T_a = \frac{1}{2}\left(m + n + b + c + d + \frac{bcd}{mn}\right)$$

Similarly,

$$T_b = \frac{1}{2}\left(m + n + a + c + d + \frac{acd}{mn}\right)$$

Hence,

$$T_a - T_b = \frac{1}{2}\left(m + n + b + c + d + \frac{bcd}{mn}\right) - \frac{1}{2}\left(m + n + a + c + d + \frac{acd}{mn}\right)$$

$$= \frac{b-a}{2} + \frac{cd(b-a)}{2mn}$$

$$= \frac{1}{2}(b-a)\left(1 + \frac{cd}{mn}\right)$$

Similarly,

$$T_b - T_c = \frac{1}{2}(c-b)\left(1+\frac{ad}{mn}\right)$$

Therefore,

$$\frac{T_a - T_b}{T_b - T_c} = \frac{\frac{1}{2}(b-a)\left(1+\frac{cd}{mn}\right)}{\frac{1}{2}(c-b)\left(1+\frac{ad}{mn}\right)}$$

$$= \frac{\frac{(b-a)(mn+cd)}{mn}}{\frac{(c-b)(mn+ad)}{mn}}$$

$$= \frac{(b-a)(mn+cd)}{(c-b)(mn+ad)}$$

Now we just have to compute $\overline{\left(\frac{T_a - T_b}{T_b - T_c}\right)}$

$$\overline{\left(\frac{T_a - T_b}{T_b - T_c}\right)}$$

$$= \frac{\left(\frac{1}{b}-\frac{1}{a}\right)\left(1+\frac{mn}{cd}\right)}{\left(\frac{1}{c}-\frac{1}{b}\right)\left(1+\frac{mn}{ad}\right)}$$

$$= \frac{\frac{(a-b)(mn+cd)}{abcd}}{\frac{(b-c)(mn+ad)}{abcd}}$$

$$= \frac{(a-b)(mn+cd)}{(b-c)(mn+ad)} = \frac{T_a - T_b}{T_b - T_c}$$

Hence, by lemma 6.3 T_a, T_b, T_c collinear.

Similarly, T_b, T_c, T_d collinear, which concludes our proof

Excerise 2.2 Find the complex equation of the Cantor Line.

By lemma 6.4 we could express the Cantor Line as

$$(\overline{T_a} - \overline{T_b})z - (T_a - T_b)\overline{z} = \overline{T_a}T_b - T_a\overline{T_b}$$

First let's compute $\overline{T_a}T_b - T_a\overline{T_b}$

$$4RHS = \left(\frac{1}{m}+\frac{1}{n}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{mn}{bcd}\right)\left(m+n+a+c+d+\frac{acd}{mn}\right)$$

$$-\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{a}+\frac{1}{c}+\frac{1}{d}+\frac{mn}{acd}\right)\left(m+nba+c+d+\frac{bcd}{mn}\right)$$

Let $\frac{1}{m}+\frac{1}{n}+\frac{1}{c}+\frac{1}{d}=X$, and $m+n+c+d=Y$, then we have

$$4RHS=\left(X+\frac{1}{b}+\frac{mn}{bcd}\right)\left(Y+a+\frac{acd}{mn}\right)-\left(X+\frac{1}{a}+\frac{mn}{acd}\right)\left(Y+b+\frac{bcd}{mn}\right)$$

$$=X\left(a+\frac{acd}{mn}-b-\frac{bcd}{mn}\right)+Y\left(\frac{1}{b}+\frac{mn}{bcd}-\frac{1}{a}-\frac{mn}{acd}\right)$$

$$+\left(\frac{1}{b}+\frac{mn}{bcd}\right)\left(a+\frac{acd}{mn}\right)-\left(\frac{1}{a}+\frac{mn}{acd}\right)\left(b+\frac{bcd}{mn}\right)$$

$$=X(a-b)\left(\frac{mn+cd}{mn}\right)+Y(a-b)\left(\frac{mn+cd}{abcd}\right)$$

$$+\frac{a(mn+cd)^2}{mnbcd}-\frac{b(mn+cd)^2}{mnacd}$$

Note that

$$\frac{a(mn+cd)^2}{mnbcd}-\frac{b(mn+cd)^2}{mnacd}=\frac{(mn+cd)^2}{mncd}\left(\frac{(a-b)(a+b)}{ab}\right)$$

Hence,

$$4RHS=(a-b)(cd+mn)\left[\frac{X}{mn}+\frac{Y}{abcd}+\frac{(mn+cd)(a+b)}{mnabcd}\right]$$

Now notice

$$\frac{X}{mn}=\frac{1}{mn}\left(\frac{1}{m}+\frac{1}{n}+\frac{1}{c}+\frac{1}{d}\right),$$

$$\frac{Y}{abcd}=\frac{m+n+c+d}{abcd},$$

$$\frac{(mn+cd)(a+b)}{mnabcd}=\frac{1}{bcd}+\frac{1}{acd}+\frac{1}{mnb}+\frac{1}{mna}$$

$$=\frac{a+b}{abcd}+\frac{1}{mn}\left(\frac{1}{a}+\frac{1}{b}\right)$$

Finally we have

$$RHS=\frac{1}{4}(a-b)(mn+cd)\left[\frac{1}{abcd}(a+b+c+d+m+n)+\frac{1}{mn}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{m}+\frac{1}{n}\right)\right]$$

Recall that

$$T_a-T_b=\frac{1}{2}(b-a)\left(\frac{mn+cd}{mn}\right),$$

$$\overline{T_a}-\overline{T_b}=\frac{1}{2}\cdot\frac{(a-b)(mn+cd)}{abcd}$$

Hence the equation of the Cantor Line could be expressed as

$$\frac{1}{2} \cdot \frac{(a-b)(mn+cd)}{abcd} z + \frac{1}{2}(a-b)\left(\frac{mn+cd}{mn}\right)\overline{z}$$

$$= \frac{1}{4}(a-b)(mn+cd)\left[\frac{1}{abcd}(a+b+c+d+m+n) + \frac{1}{mn}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{m}+\frac{1}{n}\right)\right]$$

Divding the common factors of both sides give

$$2mnz + 2abcd\overline{z} = mn(a+b+c+d+m+n) + abcd\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{m}+\frac{1}{n}\right)$$

Comments: As one could observe the algebraic workload is quite immense, hence I strongly recommend readers actually doing the algebra themselves if they have not done so.

3 Cantor's Point

(Cantor's Point) A, B, C, D, M, N, L lies on a circle. Prove that the Cantor Lines of $M, L; N, M; L, M$ with respect to quadrilateral $ABCD$ are concurrent.

Using the results we've proven above we could express the Cantor Lines of $M, L; N, M; L, M$ with respect to quadrilateral $ABCD$ as the following

$$2mnz + 2abcd\overline{z} = mn(\sum a + m + n) + abcd\left(\sum \frac{1}{a} + \frac{1}{m} + \frac{1}{n}\right) \tag{1}$$

$$2nlz + 2abcd\overline{z} = nl(\sum a + n + l) + abcd\left(\sum \frac{1}{a} + \frac{1}{n} + \frac{1}{l}\right) \tag{2}$$

$$2lmz + 2abcd\overline{z} = lm(\sum a + l + m) + abcd\left(\sum \frac{1}{a} + \frac{1}{l} + \frac{1}{m}\right) \tag{3}$$

Where $\sum a = a + b + c + d, \sum \frac{1}{a} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$.

Subtrating (2) from (1) gives

$$\begin{aligned} 2n(m-l)z &= mn(\sum a + m + n) - nl(\sum a + n + l) + abcd\left(\frac{1}{m} - \frac{1}{l}\right) \\ &= n(m-l)\sum a + mn(m+n) - nl(n+l) - abcd\left(\frac{m-l}{ml}\right) \\ &= n(m-l)\sum a + n(m^2 - l^2 + mn - nl) - abcd\left(\frac{m-l}{ml}\right) \\ &= n(m-l)\sum a + n(m-l)(m+n+l) - abcd\left(\frac{m-l}{ml}\right) \end{aligned}$$

Dividing both sides by $2n(m-l)$ gives the intersection of the Cantor Lines (1) and (2) as

$$z_0 = \frac{\sum a + m + n + l}{2} - \frac{abcd}{mnl}$$

Similarly subtracting (3) from (2) would give us

$$z_1 = \frac{\sum a + n + l + m}{2} - \frac{abcd}{nlm} = z_0$$

Hence, the three Cantor Lines intersect at a point, known as the Cantor Point.

4 Afterword

I hope you have enjoyed this article. If you have any questions or I have made any mistakes (I am after all just a maths enthusiast), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at *Prime Pursuit* for more articles and monthly problems!

5 Appendix

The proofs of all the lemmas here are left as exercises for the reader.

Lemma 6.1 The foot of altitude from Z to $\overline{AB}$ is given by

$$\frac{(\overline{a}-\overline{b})z+(a-b)\overline{z}+\overline{a}b-a\overline{b}}{2(\overline{a}-\overline{b})}$$

Lemma 6.2 A, B, C, D cyclic if and only if

$$\frac{\frac{a-c}{b-c}}{\frac{a-d}{b-d}} = (a, b, c, d) \in \mathbb{R}$$

for any permutations of (a, b, c, d)

Lemma 6.3 A, B, C collinear if and only if

$$\frac{c-a}{b-a} \in \mathbb{R} \Leftrightarrow \frac{c-a}{b-a} = \overline{\left(\frac{c-a}{b-a}\right)}$$

Lemma 6.4 The set of points z passing through point A with direction vector α is given by

$$\overline{\alpha}z - \alpha\overline{z} = \overline{\alpha}a - \alpha\overline{a}$$

Moreover the set of points z passing through points A, B is given by

$$(\overline{a}-\overline{b})z-(a-b)\overline{z} = \overline{a}b-a\overline{b}$$