Generating Functions

Authored by Jay Zhao, Edited by Michael Ma and George Zhu

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The technique of generating functions is surprisingly useful and has applications in discrete mathematics, allowing us to easily manipulate and work with sequences of numbers by treating them as functions.

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1 The Generating Function

The ordinary generating function for the sequence $(a_0, a_1, a_2, a_3, \dots)$ is the formula power series

$$F(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

We say that it is a "formal" power series because we usually don't really care what x actually is. It's a dummy variable so we usually do not need to worry about if this function actually converges.

The pattern here is that the i-term is the coefficient of the x^i term of the generating function.

We'll use the notation $(a_0, a_1, a_2, ...) \longleftrightarrow F$ to indicated that F is the generating function of the sequence $(a_0, a_1, \dots)$.

Examples

$$(1,2,3,4,5) \longleftrightarrow 1 + 2x + 3x^2 + 4x^3 + 5x^4$$
$$(1,3,5,7,9) \longleftrightarrow 1 + 3x + 5x^2 + 7x^3 + 5x^4$$

1.1 Why Generating Functions?

We might wonder what the purpose of turning our sequence into a generating function is. It seems that all we have done is add unnecessary x^is to our sequence.

Consider the sequence $(1, 1, 1, 1, \dots)$

$$(1,1,1,\dots) \longleftrightarrow 1 + x + x^2 + \dots$$

I claim that:

$$(1,1,1,\dots) \longleftrightarrow \frac{1}{1-x}$$

The reader should try to prove this for themselves or at least convince themselves that this is true.

Remark 1. The calculus scholars among you might recognize this as $1 + x + x^2 + \dots$ being the Taylor series expansion of $\frac{1}{1-x}$.

You might also recognize that $1 - x^n = (1-x)(1 + x + \cdots + x^{n-1})$.

Below I present an intuitive proof of the above claim.

$$\begin{aligned} 1 + x + x^2 + x^3 + \dots &= 1 + (x + x^2 + x^3 + x^4 + \dots) \\ &= 1 + x(1 + x + x^2 + x^3 + \dots) \\ (1-x)(1 + x + x^2 + x^3 + \dots) &= 1 \\ 1 + x + x^2 + x^3 + \dots &= \frac{1}{1-x} \end{aligned}$$

Although the sequence $1 + x + x^2 + x^3 + \dots$ does not actually converge when $|x| \geq 1$, as aforementioned, we don't have to worry about convergence.

More generally

$$(a, ar, ar^2, ar^3, ar^4 \dots) \longleftrightarrow \frac{a}{1 - rx}.$$

The proof of this is left as an exercise to the reader.

We have managed to convert a set of infinite integer sequence into a simple function! This should hopefully give some sort of motivation for why generating functions are useful.

2 Manipulating Generating Functions

2.1 Arithmetic

Scaling The most simple operation we can perform on a sequence is to scale all its terms by a factor k.

$$\begin{aligned} &(a_0, a_1, \dots) \longleftrightarrow F(x) \\ \iff &(ka_0, ka_1, \dots) \longleftrightarrow kF(x) \end{aligned}$$

This is because multiplying $F(x)$ by k is equivalent to multiplying each term by k.

Adding Two Generating Functions Another way we can manipulate generating functions is by adding two of them together like how we did to prove that $1 + x + x^2 + \dots = \frac{1}{1 - x}$.

Adding two generating functions together involves adding up the x^i terms of teach function.

2.2 Shifting

Shifting a generating function over involves multiplying by a factor of x.

If we had

$$F(x) = a_0 + a_1 x + \dots$$

then

$$xF(x) = a_0 x + a_1 x^2 + \dots$$

This is equivalent to converting the sequence from $(a_0, a_1 \dots)$ to $(0, a_0, a_1, \dots)$.

2.3 Sum and Difference

Lets say we wanted to create a sequence B based on the sequence A where

$$b_n = \sum_{k=0}^{n} a_k.$$

Let $A \longleftrightarrow F(x)$,

$$B \longleftrightarrow a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots$$

$$\begin{aligned}
&a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \dots \\
&= a_0(1 + x + x^2 + \dots) + a_1 x + (a_1 + a_2)x^2 + \dots \\
&= a_0(1 + x + \dots) + a_1(x + x^2 + \dots) + a_2(x^2 + x^3 + \dots) + \dots \\
&= \frac{a_0}{1-x} + \frac{a_1 x}{1-x} + \frac{a_2 x^2}{1-x} + \dots \\
&= \frac{a_0 + a_1 x + a_2 x^2 + \dots}{1-x} \\
&= \frac{F(x)}{1-x}.
\end{aligned}$$

If we set A to be the sequence $(1, 1, 1, 1, 1, \dots)$, then

$$\frac{1}{(1-x)^2} = \frac{F(x)}{1-x} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

Remark 2. You might notice that the derivative of $\frac{1}{1-x}$ is actually $\frac{1}{(1-x)^2}$ and there is indeed a reason for this.

If we take the "sum function" of $\frac{1}{(1-x)^2}$ we obtain the sequence of triangle numbers, as this transforms $(1, 2, 3, 4, \dots) \to (1, 1+2, 1+2+3, \dots)$.

Some Questions to think about

Knowing that the nth triangle number is given by $\frac{n(n+1)}{2}$, can we find the generating function for the sequence of square numbers? This is left as an exercise to the reader.

We can also multiply by $(1-x)$ instead of dividing by it. What effect might this have on the sequence?

2.4 Differentiation and Integration

Knowing our generating function is a power series, we can differentiate our function term by term using the power rule.

$$\frac{d}{dx}x^n = nx^{n-1}$$

Notice that for the *nth* term, we transform it by multiplying by n and shifting it down by 1

Overall this has the effect of

$$(a_0, a_1, a_2, \dots) \to (1a_1, 2a_2, 3a_3, \dots) \to (2a_2, 6a_3, 12a_4 \dots)$$

And knowing our generating function is also a function, we can differentiate it as a whole.

Seeing the coefficients of $1, 2, 3, \dots$ might remind us of our previous example. Consider the function

$$A \longleftrightarrow F(x) = \frac{1}{1-x}$$

then we can see that

$$F'(x) = 1 + 2x + 3x^2 + \dots$$

Integration is very similar, but instead of us multiplying the ith term by i and shifting down, the effect is shifting up and then multiplying the ith term by i.

We can see that integration and differentiation are inverses.

Remark 3. When integration, there is a constant of integration which one might need to consider when integrating a generating function.

Manipulation Exercises Determine the generating functions for the following sequences:

1. $(1, -1, 1, -1, 1, -1, \dots)$
2. $(1, 0, 1, 0, 1, 0, \dots)$
3. $\left(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots\right)$

3 The Fibonacci Sequence

The Fibonacci numbers are defined with the following recursive formula:

$$\begin{aligned} F_0 &= 0 \\ F_1 &= 1 \\ F_n &= F_{n-1} + F_{n-2} \end{aligned}$$

Consider $(F_0, F_1, \dots) \longleftrightarrow F$ Using our recursive formula we can see that

$$\begin{array}{cccccc} +(& 0, & 1, & 0, & 0, & \dots &) \\ +(& 0, & 0, & F_0, & F_1, & \dots &) \\ +(& 0, & F_0, & F_1, & F_2, & \dots &) \\ \hline (& F_0, & F_1, & F_2, & F_3, & \dots &) \end{array}$$

which is equivalent to

$$x + xF(x) + x^2F(x) = F(x)$$

which can be rearranged to give

$$\begin{aligned}F(x) &= \frac{x}{1-x-x^2}\\ &= \frac{x}{\left(1-\frac{1+\sqrt{5}}{2}x\right)\left(1-\frac{1-\sqrt{5}}{2}x\right)}\\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}x}-\frac{1}{1-\frac{1-\sqrt{5}}{2}x}\right).\end{aligned}$$

Finally we can use the $\dfrac{a}{1-rx}$ formula to determine the Fibonacci sequence, which is given by

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

We've now re derived this famous formula for ourselves and in fact this method can be applied to solve and linear recursive formula.

Remark 4. Some might know of a formula for this type of equation which involves the *roots of the characteristic equation*, This comes from our partial fractions. It should also hopefully explain the rule regarding repeated roots which also comes from our partial fractions.

3.1 AMO 2020 Problem 4

We will now take a look at Question 4 From 2020's Australian Maths Olympiad

> Define the sequence $A_1, A_2, A_3, \dots$ by $A_1 = 1$ and for $n = 1, 2, 3, \dots$
>
> $$A_{n+1} = \frac{A_n + 2}{A_n + 1}$$
>
> Define the sequence $B_1, B_2, B_3, \dots$ By $B_1 = 1$ and for $n = 1, 2, 3, \dots$
>
> $$B_{n+1} = \frac{B_n^2 + 2}{2B_n}$$
>
> Prove that $B_{n+1} = A_{2^n}$ for all non-negative integers n.

Our method of proof will be to determine a closed form formula for A_n and then use induction to show that $B_{n+1} = A_{2^n}$.

if we let $A_n = \dfrac{p}{q}$ then

$$A_{n+1} = \frac{\frac{p}{q}+2}{\frac{p}{q}+1} = \frac{p+2q}{p+q}.$$

Let $P_1, P_2, P_3, \dots$ and $Q_1, Q_2, Q_3, \dots$ be sequences such that $P_1 = 1$ and $Q_1 = 1$ and for $n = 1, 2, 3, \dots$

$$P_{n+1} = P_n + 2Q_n$$

$$Q_{n+1} = P_n + Q_n.$$

We have that $A_n = \dfrac{P_n}{Q_n}$ for all n.

Let $(P_1, P_2, P_3, \dots) \longleftrightarrow P(x)$ and $(Q_1, Q_2, Q_3, \dots) \longleftrightarrow Q(x)$. Using this we can form the equations

$$\begin{aligned} xP(x) + 2xQ(x) &= P(x) - 1 \\ xP(x) + xQ(x) &= Q(x) - 1 \end{aligned}$$

Or

$$(x-1)P(x) + 2xQ(x) = -1 \tag{1}$$
$$xP(x) + (x-1)Q(x) = -1 \tag{2}$$

We can use Crammers Rule to solve for $P(x)$ and $Q(x)$:

$$\begin{aligned} P(x) &= \frac{2x - (x-1)}{(x-1)^2 - 2x^2} = \frac{x+1}{-x^2 - 2x + 1} \\ Q(x) &= \frac{x - (x-1)}{(x-1)^2 - 2x^2} = \frac{1}{-x^2 - 2x + 1} \end{aligned}$$

Now we can turn these into partial fractions.

$$\begin{aligned} Q(x) &= \frac{1}{2\sqrt{2}} \left(\frac{1}{-1+\sqrt{2}-x} - \frac{1}{-1-\sqrt{2}-x} \right) \\ P(x) &= \frac{1}{2} \left(\frac{1}{-1+\sqrt{2}-x} + \frac{1}{-1-\sqrt{2}-x} \right) \end{aligned}$$

Notice that

$$\frac{1}{a-x} = \frac{\frac{1}{a}}{1-\frac{x}{a}} = \frac{1}{a} + \frac{x}{a^2} + \frac{x^2}{a^3} + \dots$$

This must mean that

$$\begin{aligned} Q_n &= \frac{1}{2\sqrt{2}} \left(\left(\frac{1}{-1+\sqrt{2}} \right)^n - \left(\frac{1}{-1-\sqrt{2}} \right)^n \right) \\ P_n &= \frac{1}{2} \left(\left(\frac{1}{-1+\sqrt{2}} \right)^n + \left(\frac{1}{-1-\sqrt{2}} \right)^n \right). \end{aligned}$$

and finally

$$A_n = \frac{P_n}{Q_n} = \frac{\frac{1}{2}\left(\left(\frac{1}{-1+\sqrt{2}} \right)^n + \left(\frac{1}{-1-\sqrt{2}} \right)^n \right)}{\frac{1}{2\sqrt{2}}\left(\left(\frac{1}{-1+\sqrt{2}} \right)^n - \left(\frac{1}{-1-\sqrt{2}} \right)^n \right)} = \sqrt{2}\frac{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n}{\left(1+\sqrt{2}\right)^n - \left(1-\sqrt{2}\right)^n}.$$

We already know that $A_1 = B_1$ we will show by induction that if $A_n = B_m$ then $A_{2n} = B_{m+1}$.

It is sufficient to show that

$$A_{2n} = \frac{A_n^2 + 2}{2A_n}.$$

$$\text{RHS} = \frac{\left(\sqrt{2}\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}\right)^2+2}{2\sqrt{2}\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}} = \frac{\frac{(1+\sqrt{2})^n+(1-\sqrt{2})^n}{(1+\sqrt{2})^n-(1-\sqrt{2})^n}+\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{(1+\sqrt{2})^n+(1-\sqrt{2})^n}}{\sqrt{2}}$$

if we let $x = \left(1+\sqrt{2}\right)^n$ and $y = \left(1-\sqrt{2}\right)^n$ we get that

$$\text{RHS} = \frac{1}{\sqrt{2}}\left(\frac{x+y}{x-y}+\frac{x-y}{x+y}\right) = \frac{1}{\sqrt{2}} \times \frac{(x+y)^2+(x-y)^2}{(x-y)(x+y)} = \sqrt{2}\frac{x^2+y^2}{x^2-y^2} = A_{2n}$$

□

4 Counting with Generating Functions

Generating functions are extremely useful in combinatorics and specifically in counting problems. We can encode the number of ways to do something in the coefficients of our generating function.

Consider a set S such that $|S| = n$. We wish to construct a generating function where the coefficient of x^i represent the number of subsets of S of size i.

It might be easier to consider a different function first. Let $S = \{a_1, a_2, a_3, ..., a_n\}$ and consider the function

$$\prod_{i=1}^{n}(1+x_i) = (1+x_1)(1+x_2)\dots(1+x_n)$$

Expanding this product will yield the sum of 2^n monic polynomials. For example,

$$\begin{aligned}(1+x_1)(1+x_2)(1+x_3) = 1&\\ +x_1+x_2+x_3&\\ +x_1x_2+x_1x_2+x_2x_3&\\ +x_1x_2x_3.&\end{aligned}$$

Each term represents a unique subset (e.g. in the above example 1, x_2, x_1x_3 represents $\emptyset$, $\{a_2\}$, $\{a_1, a_3\}$ respectively). However, we only care about the size of each subset, which is simply the degree of each term of the expression. Therefore we can treat $x_1, x_2, \dots, x_n$ as indistinguishable, leading to our final generating function $(1+x)^n$.

The generating function also represents the number of ways of choosing i items without replacement. The x^i term has coefficient $\binom{n}{i}$ by the binomial theorem.

Now consider the number of ways of choosing items with replacement. This means that now it is possible to choose the same element more than once (resulting in a multiset). Thus our $(1+x)$ term gets replaced with $(1+x+x^2+x^3+\dots) = \frac{1}{1-x}$. And so the generating function for the number of ways of choosing from n items with replacement is

$$\left(\frac{1}{1-x}\right)^n = \frac{1}{(1-x)^n}.$$

We can easily calculate the coefficients of the terms using Taylor Series. We know that for a given function:

$$f(x) = f(0) + f'(0)x + f''(0)x^2 + f'''(0)x^3 + \dots$$

The details of this are left for the reader.

4.1 Partitioning Integers with Generating Functions

Consider the number of multisets of natural numbers which add to n and let this be $p(n)$, we wish to find a generating function which represents the number of *partitions* of n.

To construct the generating function, we first consider how many times a specific positive integer k appears in the partition. This choice is represented by the function $1 + x^k + x^{2k} + x^{3k} + \dots = \frac{1}{1-x^k}$, as choosing the integer k a total of $0, 1, 2, 3, \dots$ times will contribute a sum of $0, k, 2k, 3k, \dots$ to the partition.

Now we just need to combine the generating functions for all positive k to get

$$\prod_{k=1}^{\infty} \frac{1}{1-x^n} = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\dots}.$$

To find $p(n)$ now, we simply use series expansion to find the coefficient of x^n.

If we only wanted partitions where the parts are all odd then we will end up with the generating function

$$\prod_{k \text{ is odd}}^{\infty} \frac{1}{1-x^n} = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\dots}.$$

as we only consider odd k in our partition.

If instead we wanted the number of partitions where each part is different, then each integer k will contribute a factor of $(1 + x^k)$ to the generating function, since each k can be chosen at most once. This gives the generating function

$$\prod_{k=1}^{\infty} (1 + x^k) = (1+x)(1+x^2)(1+x^3)\dots$$

There is actually an interesting result. The number of partitions of an integer n into distinct parts is equal to the number of n into odd parts. To prove this we will have to prove that

$$(1+x)(1+x^2)(1+x^3)\dots = \frac{1}{(1-x)(1-x^3)(1-x^5)}$$

$$\text{LHS} = \frac{\cancel{1-x^2}}{1-x} \times \frac{\cancel{1-x^4}}{\cancel{1-x^2}} \times \frac{\cancel{1-x^6}}{1-x^3} \times \frac{\cancel{1-x^8}}{\cancel{1-x^4}} \times \dots$$

We can cancel all the terms with an even power of x and be left with only odd powers of x on the bottom which indeed does give us our desired result.

4.2 Generating Functions and Complex Numbers

We will now take a look at the following problem which I first saw in a 3Blue1Brown video.

> Find the number of subsets of $\{1, 2, 3, 4, 5, \ldots, 2000\}$, the sum of whose elements is divisible by 5.

We can recognize that the number of subsets whose elements sum to n can be found using the generating function

$$F(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)\ldots(1+x^{2000}).$$

Let

$$F(x) = A_0 + A_1x + A_2x^2 + \cdots + A_{2001000}x^{2001000}.$$

We are only interested in the terms with x to the power of a multiple of 5 in fact we wish to somehow get $A_0 + A_5 + A_{10} + \ldots$ Which requires $x^5 = 1$

This inspires us to consider the roots of unity. Let $1, \zeta_1, \zeta_2, \zeta_3, \zeta_4$ be our roots of unity.

$$F(1) + F(\zeta_1) + F(\zeta_2) + F(\zeta_3) + F(\zeta_4) + F(\zeta_5) = 5(A_0 + A_5 + \ldots)$$

$$5(A_0 + A_5 + \ldots) = (1+1)^{2000} + 4\left((1+\zeta_1)(1+\zeta_2)(1+\zeta_3)(1+\zeta_4)(1+1)\right)^{400}$$

$$A_0 + A_5 + \cdots = \frac{1}{5}(2^{2000} + 4 \cdot 2^{400})$$

5 Problems

1. In how many ways can we fill a bag with n fruits subject to the following constraints?
   - The number of apples must be even.
   - The number of bananas must be a multiple of 5.
   - There can be at most four oranges.
   - There can be at most one pear.
2. **(2021 BMO Round 1 P3)** For each pile of integer $0 \leq n \leq 11$, Eliza has exactly three identical pieces of gold that weigh 2^n grams. In how many different ways can she form a pile of gold weighing 2021 grams?
3. **(1994 China MO Day 2, P5)** Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}.$$

4. **(2022 IMC Day 1, P3)** Let p be a prime number. A flea is staying at point 0 of a real line. At each minute, the flea has three possibilities: to stay at its position, or to move by 1 to the left or to the right. After $p-1$ minutes, it wants to be at 0 again. Denote by $f(p)$ the number of its strategies to do this (for example, $f(3) = 3$: it may either stay at 0 for the entire time, or go to the left then to the right, or go to the right and then to the left. Find $f(p)$ modulo p.

6 Afterword

I hope you have enjoyed this article. If you have any questions or I have made any mistakes (I am after all just a maths enthusiast), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at *Prime Pursuit* for more articles and monthly problems!