Monthly Problems 2

February 2024

Contact Us: *primusmathematica1729@gmail.com*
Website: *Prime Pursuit*
Youtube: *Primus Mathematica*

Problems

1. For which positive integers $n \geq 3$ does there exist a permutation $(a_1, a_2, ..., a_n)$ of $(1, 2, \ldots, n)$ such that $|a_i - a_{i-1}|$ and $|a_{i+1} - a_i|$ differ in parity for all $i = 2, 3, ..., n$, where $a_{n+1} = a_1$? *(Alston Yam)*

2. Let A be a point lying strictly inside the circle ω. Find the locus of all points M such that M is the midpoint of a chord XY of ω and $\angle XAY = 90^\circ$. *(Culver Kwan)*

3. Suppose a is a quadratic residue for all but finitely many primes. Prove that a is a perfect square. *(Culver Kwan)*

   (For any positive integer p, an integer a is a *quadratic residue* (mod p) if there exists an integer b such that $p \mid a - b^2$.)

4. Let $n \geq 3$ be a positive integer, and set $N = \dfrac{n(n-1)}{2}$. Find the largest possible positive integer m, such that for any (possibly empty) subsets $A_1, A_2, ..., A_n$ of the set $\{1, 2, ..., N\}$, whenever $|A_i \setminus A_j| \leq m$ for all positive integers i, j with $1 \leq i, j \leq n$, the coefficient of the term $\prod_{i=1}^{N} x_i$ in

$$\prod_{1 \leq i < j \leq n} \left(\sum_{k \in A_i} x_k - \sum_{k \in A_j} x_k \right)$$

   is equal to zero. *(Culver Kwan)*