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February Monthly Problem Set 1 Solutions

Authored by Alston Yam

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In this article we present the solutions to the first February Problem Sheet.

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1 Problems

1.1 Problem 1

Find all triplets of positive integers (a, b, c) such that

$$\frac{\ln(ab + bc + ac)}{\ln(\text{lcm}(a, b, c))}$$

is an integer. (*Culver Kwan*)

Solution. We claim that the only triples (a, b, c) that satisfy the given requirement are:

$$(a, b, c) = (3, 3, 3),$$

$$(a, b, c) = \{1, 2, 2\},$$

$$(a, b, c) = \{2d, 3d, 6d\}, \text{ where } d \text{ is any positive integer.}$$

Note that for example, $(a, b, c) = \{1, 2, 2\}$ represents all permutations of the set $\{1, 2, 2\}$.

It is not hard to show that they all satisfy the given condition.

We now prove that these are the only solutions. Let

$$\frac{\ln(ab + bc + ac)}{\ln(\text{lcm}(a, b, c))} = k,$$

Where k is a positive integer.

Let

$$l = \text{lcm}(a, b, c),$$

Note that

$$\begin{aligned} &\iff \ln(ab + bc + ac) = k\ln(l) \\ &\iff e^{\ln(ab+bc+ac)} = e^{k\ln(l)} = (e^{\ln(l)})^k \\ &\iff ab + bc + ac = l^k. \end{aligned}$$

Lemma:

$$k \leq 3.$$

Proof: WLOG let $a \geq b \geq c$.

$$a^k | l^k = ab + bc + ac$$

$$\implies a^k \leq ab + bc + ac \leq 3ab$$

$$\implies a^{k-1} \leq 3b \leq 3a$$

$$\implies a^{k-2} \leq 3 \quad (\spadesuit)$$

FTSOC let $k \geq 4$:

$$a^2 \leq a^{k-2} \leq 3$$

$$\implies a \leq \sqrt{3}$$

$$\implies a = b = c = 1$$

Substituting back, we get a contradiction; hence the lemma is proven.

We now do case work on k:

Case 1: $k = 1$

$$l = ab + bc + ac$$

WLOG $a \geq b \geq c$,

Since $a|l$, $a|LHS \implies a|RHS$.

$$a|ab + bc + ac \implies a|bc.$$

Since

$$a, b, c|bc$$

Therefore

$$l|bc$$

$$\implies l \leq bc$$

But $l = ab + bc + ac \geq 3bc$,

$$l \geq 3bc$$

Contradiction!

Case 2: $k = 2$

$$l^2 = ab + bc + ac$$

Define $d = \gcd(a, b, c)$, $d \geq 1$.

Let

$$a = dA, b = dB, c = dC$$

where there exists two letters, WLOG A, B, such that $\gcd(A, B) = 1$.

Note that

$$\operatorname{lcm}(a, b, c) = d \operatorname{lcm}(A, B, C).$$
$$d^2 \operatorname{lcm}(A, B, C)^2 = d^2 AB + d^2 BC + d^2 AC$$
$$\operatorname{lcm}(A, B, C)^2 = AB + BC + AC$$

and since $A | \operatorname{lcm}(A, B, C)$, so $A | AB + BC + AC$, so $A | BC$. Similarly, $B | AC$ and $C | AB$.

$\because \gcd(A, B) = 1$, $\therefore A | C$ and $B | C$, $\implies AB | C$. and since $C | AB$,

$$AB \leq C \leq AB$$

$$\therefore C = AB.$$

so

$$\operatorname{lcm}(A, B, C) = AB$$
$$A^2 B^2 = AB + A^2 B + B^2 A$$
$$AB = 1 + A + B$$
$$\implies A | (B + 1), B | (A + 1)$$

If $A = B + 1$, $\implies (A - 1) | (A + 1) \implies A = 2$ or 3. Substitute back and we get $B = 1$ or 2. The case where $B = 1$ gives $(A, B, C) = \{2, 1, 2\}$, however, substituting back we get that $k = 3$ so we discard this solution here. When $B = 2$ we get $(A, B, C) = \{3, 2, 6\}$ which indeed works. So

$$(a, b, c) = \{3d, 2d, 6d\}$$

For some integer d is a solution to the original equation.

Now, if $A \neq B + 1$, we get $A \leq \dfrac{B + 1}{2}$, and since $B \leq A + 1$ we get $B \leq \dfrac{B + 1}{2} + 1$, $\implies B \leq 3$. Case work gives solutions which we've already covered before. Hence

$$(a, b, c) = \{3d, 2d, 6d\}$$

are the only solutions in this case.

Case 3: $k = 3$

From (♠),

$$a^{3-2} = a \leq 3$$

and case work on a, b, c gives solutions

$$(a, b, c) = \{1, 2, 2\}$$
$$(a, b, c) = (3, 3, 3)$$

Altogether, we get that

$$(a, b, c) = (3, 3, 3)$$
$$(a, b, c) = \{2, 2, 1\}$$
$$(a, b, c) = \{3d, 2d, 6d\}$$

are the only solutions. □

1.2 Problem 2

x, y, z, t are positive reals summing to 4. Prove that

$$\frac{x}{1+y^2} + \frac{y}{1+z^2} + \frac{z}{1+t^2} + \frac{t}{1+x^2} \geq 2$$

(Culver Kwan)

Solution. We first start with a claim:

Lemma:

$$2 \geq \frac{xy^2}{1+y^2} + \frac{yz^2}{1+z^2} + \frac{zt^2}{1+t^2} + \frac{tx^2}{1+x^2}$$

Proof: We proceed by AM-GM inequality.

AM-GM on x and $\frac{xy^4}{(1+y^2)^2}$ gives

$$\frac{x + \frac{xy^4}{(1+y^2)^2}}{2} \geq \sqrt{\frac{x^2y^4}{(1+y^2)^2}} = \frac{xy^2}{1+y^2}$$

$$\frac{x}{2} + \frac{xy^4}{2(1+y^2)^2} \geq \frac{xy^2}{1+y^2}$$

Summing cyclically over all four inequalities:

$$\frac{x}{2} + \frac{xy^4}{2(1+y^2)^2} + \frac{y}{2} + \frac{yz^4}{2(1+z^2)^2} + \frac{z}{2} + \frac{zt^4}{2(1+t^2)^2} + \frac{t}{2} + \frac{tx^4}{2(1+x^2)^2} \geq \frac{xy^2}{1+y^2} + \frac{yz^2}{1+z^2} + \frac{zt^2}{1+t^2} + \frac{tx^2}{1+x^2}$$

Substitute $x + y + z + t = 4$:

$$2 + \frac{xy^4}{2(1+y^2)^2} + \frac{yz^4}{2(1+z^2)^2} + \frac{zt^4}{2(1+t^2)^2} + \frac{tx^4}{2(1+x^2)^2} \geq \frac{xy^2}{1+y^2} + \frac{yz^2}{1+z^2} + \frac{zt^2}{1+t^2} + \frac{tx^2}{1+x^2}$$

and since

$$\frac{xy^4}{2(1+y^2)^2} + \frac{yz^4}{2(1+z^2)^2} + \frac{zt^4}{2(1+t^2)^2} + \frac{tx^4}{2(1+x^2)^2} \geq 0,$$

the lemma is proven.

Now from the lemma,

$$4 \geq \frac{xy^2}{1+y^2} + \frac{yz^2}{1+z^2} + \frac{zt^2}{1+t^2} + \frac{tx^2}{1+x^2} + 2$$

$$x - \frac{xy^2}{1+y^2} + y - \frac{yz^2}{1+z^2} + z - \frac{zt^2}{1+t^2} + t - \frac{tx^2}{1+x^2} \geq 2$$

$$\frac{x}{1+y^2} + \frac{y}{1+z^2} + \frac{z}{1+t^2} + \frac{t}{1+x^2} \geq 2$$

As desired. □

1.3 Problem 3

Let P be a point on the circumcircle of triangle ABC with circumcenter O. Let G_1, G_2, G_3 be the centroid of triangles PBC, PAC, PAB respectively. Let K be the first intersection of the circumcircle of triangle $G_1G_2G_3$ and the median of BC with respect to A. Let H', O' be the orthocenter and the circumcenter of $G_1G_2G_3$ respectively. Prove that $O'H' = KO$. (George Zhu)

Solution. We first prove that K is the centroid of triangle ABC by reverse reconstruction.

Let K' be the centroid of triangle ABC. By definition, K' lies on the A-median of triangle ABC. It remains to show that $K'G_1G_2G_3$ lies on the same circle.

Consider the midpoint M of side BP. By definition, MG_2A and MG_3C are collinear. Note that due to the ratio that the centroid splits the median into is $1:2$, we get that

$$\frac{MG_2}{G_2A} = \frac{MG_3}{G_3C} = \frac{1}{2}.$$

In other words, $G_2G_3 \parallel AC$.

Similarly, $G_3G_1 \parallel BA$, $G_1K' \parallel BP$, and $G_2K' \parallel CP$.

Thus there exists a homothety h that maps $K'G_1G_3G_2$ to $ACPB$. And since $ACPB$ cyclic, we get that $K'G_1G_3G_2$ must also be cyclic. Hence $K' = K$, and we have proven that K is indeed the centroid of triangle ABC.

We now prove that the homothety h mentioned earlier has a scale factor of -3. This is apparent considering

$$\frac{AC}{G_1G_2} = \frac{MA}{MG_2} = 3.$$

So the homothety that takes AC to G_1G_2 must have a scale factor of -3 (the negative sign is due to the fact that G_2G_3 and AC lie on opposite sides of the homothety center).

Denote the orthocenter of triangle ABC by H. Homothety h takes O' to O, and H' to H, where $OH = 3O'H'$.

Note that since K is the centroid of triangle ABC, we get that H, K, O are collinear on the Euler line of triangle ABC. Furthermore, by the ratio on the Euler lines, we have

$$3KO = OH.$$

Combine that with our previous result, we get that

$$3O'H' = 3KO,$$

$$O'H' = KO$$

As desired. □

1.4 Problem 4

Let n be a positive integer. Culver and George are playing a game using n piles of stones, initially with $1, 2, ..., n$ stones in each of the piles respectively. Culver and George take turns to play, with Culver starting first. In a turn, a player chooses a pile with a positive number of stones remaining and discards 4^k stones from the pile where k is a non-negative integer and 4^k does not exceed the number of stones in the pile before the move. Whoever discards the last stone wins. If both of the players play optimally, for which n will George win? *(Culver Kwan)*

Solution.

Claim:

For

$$n \equiv 0, 3, 9 \pmod{10},$$

George wins the game.

We first explore a simplified version of the problem: consider the game where there is only one pile of n coins, and Culver and George take turns to remove 4^k coins from the pile.

Let's call this game $(4^k) - nim$. As we will see, the original game is simply composed of n piles of $(4^k) - nim$.

Since we want the second player (George) to win, we would want to determine which starting positions are losing positions.

Calculating the Grundy numbers for the first few positions of $(4^k) - nim$, we get:

Amount of stones in the pile:	0	1	2	3	4	5	6	7	8	9	10	...
Grundy number of this position:	0	1	0	1	2	0	1	0	1	2	0	...

So it seems that when the amount of stones in the pile is congruent to $0 \pmod 5$ or $2 \pmod 5$, the Grundy number of the position is 0, which means that the position is a losing one. Hence if the pile started with this many stones, George would be the winner.

We now have the following lemma:

Lemma 1: The sequence $0, 1, 0, 1, 2$ is periodic in the Grundy numbers of the positions of $(4^k) - nim$.

Proof: We go by induction. Let $g(n)$ denote the Grundy number for when there are n stones left in a game of $(4^k) - nim$. We claim :

$$g(n) = \begin{cases} 0 & \text{for } n \equiv 0, 2 (\text{mod} 5) \\ 1 & \text{for } n \equiv 1, 3 (\text{mod} 5) \\ 2 & \text{for } n \equiv 4 (\text{mod} 5) \end{cases}$$

Base Case: Consider the table above.

Now assume that for all $n \leq 5m - 1$ for some m, the claim holds.

We now consider $\{5m, 5m+1, 5m+2, 5m+3, 5m+4\}$.

By taking off 4^k stones from the pile, we are changing the value of n under mod 5 by $+1$ or -1, as

$$4^k \equiv \pm 1 (\text{mod } 5).$$

Hence the Grundy numbers for the following are: $\begin{cases} g(5m) = mex\{1,2\} = 0 \\ g(5m+1) = mex\{0,0\} = 1 \\ g(5m+2) = mex\{1,1\} = 0 \\ g(5m+3) = mex\{0,2\} = 1 \\ g(5m+4) = mex\{0,1\} = 2 \end{cases}$

And so our lemma is proven.

Returning to our original problem, let G denote the game. We see that G is consisted of n games of $(4^k)-nim$, each having a Grundy number. We denote the *nim-sum* to be the xor of all the Grundy numbers in each of the individual piles. For example: $n = 3$, we have piles of sizes

$$(1, 2, 3)$$

So we have Grundy numbers

$$(1, 0, 1).$$

Taking the xor, we get that the *nim-sum* is 00_2, where 00_2 is in base 2.

Note that since the Grundy numbers in this game have values $0, 1, 2$, the possible values of *nim-sum* in this case are $00_2, 01_2, 10_2, 11_2$.

Lemma 2: Let P be a position in G. if the *nim-sum* of P is equal to 00_2, P is a losing position. Otherwise, P is a winning position.

Proof: We split the proof into three parts:

We first see that the final (L) position,

$$(0, 0, \dots 0)$$

Has a *nim-sum* of 00_2.

We now show two things;

- From a position where the *nim-sum* is 00_2, all moves will lead to a position where the *nim-sum* $\neq 00_2$.
- It is always possible to change the *nim-sum* from a number that is $\neq 00_2$ to 00_2.

Firstly: Assume we have a position P where the *nim-sum* is 00_2. Therefore when all the Grundy numbers of the individual piles are arranged in a table, each column has an even amount of 1s. When a move is done on any pile, the Grundy number of that pile is changed, and hence the parity of the total amount of 1s on some columns are changed, thus making the *nim-sum* not zero.

Secondly: Assume that the *nim-sum* of the current position $\neq 00_2$. Note that the $nim{-}sum$ in this case can only be $01_2, 10_2, 11_2$ as we have assumed it to not equal to 00_2. Note that the Grundy numbers $0, 1, 2$ have binary representation of $00_2, 01_2, 10_2$ respectively.

Case 1: *nim-sum* $= 01_2$:

Hence there is an odd amount of 1s in the right most column. Thus there must exist a single pile where its Grundy number is 1. Changing the Grundy number of that pile to 0 by removing some stones from it (it is always possible as by the definition of the Grundy numbers), we have flipped the parity of the amount of 1s in the right-most column, so we have changed the *nim-sum* to 00_2

Case 2: *nim-sum* $= 10_2$:

Hence there must exist a singular pile whose Grundy number is 2. By similar argument to case 1, we can change the *nim-sum* of this position to 00_2.

Case 3: *nim-sum* $= 11_2$:

Therefore there must exist two piles, one with a Grundy value of 1 and another with a Grundy value of 2. Now, we switch the pile with Grundy value 2 to the pile with Grundy value 1 (This is always possible by simply removing 1 stone from such a pile). And as a result, we have affected the parity for both columns and thus the *nim-sum* is now 00_2.

Finally, when the starting position of G has a *nim-sum* of 00_2, The position is a losing position for Culver, and hence George will win.

Writing out the Grundy values for $1, 2, 3, \dots$, We get that for $n \equiv 0, 3, 9 \pmod{10}$, the $nim - sum$ of piles $(1, 2, 3, \dots n)$ are 00_2. Thus for these values of n, the starting position is a losing one for Culver, so George should be able to force a win. $\square$

2 Afterword

I hope you have enjoyed this article. If you have any questions or I have made any mistakes (I am after all just a maths enthusiast), feel free to email to primusmathematica1729@gmail.com. Check us out on Youtube, and stay tuned at *Prime Pursuit* for more articles and monthly problems!