Monthly Problems 1

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Problems

1. Find all triplets of positive integers (a, b, c) such that

$$\frac{\ln(ab + bc + ac)}{\ln(\text{lcm}(a, b, c))}$$

is an integer. *(Culver Kwan)*

2. x, y, z, t are positive reals summing to 4. Prove that

$$\frac{x}{1+y^2} + \frac{y}{1+z^2} + \frac{z}{1+t^2} + \frac{t}{1+x^2} \geq 2.$$

(Culver Kwan)

3. Let P be a point on the circumcirle of $\triangle ABC$ with circumcenter O. Let G_1, G_2, G_3 be the centroid of PBC, PAC, PAB respectively. Let K be the first intersection of the circumcircle of $G_1G_2G_3$ and the median of BC with respect to A. Let H', O' be the orthocenter and the circumcenter of $\triangle G_1G_2G_3$ respectively. Prove that $O'H' = KO$. *(George Zhu)*

4. Let n be a positive integer. Culver and George are playing a game using n piles of stones, initially with $1, 2, ..., n$ stones in each of the piles respectively. Culver and George take turns to play, with Culver starting first. In a turn, a player chooses a pile with a positive number of stones remaining and discards 4^k stones from the pile where k is a non-negative integer and 4^k does not exceed the number of stones in the pile before the move. Whoever discards the last stone wins. If both of the players play optimally, for which n will George win? *(Culver Kwan)*